Math 53: Multivariable Calculus **Sections 201, 204**

Worksheet for 2021-10-27

Conceptual questions

Question 1. Let C be an oriented curve, and $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ a vector field. The "work" line integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ can be rewritten as

$$\int_C P(x, y)\,\mathrm{d}x + \int_C Q(x, y)\,\mathrm{d}y$$

or, letting $\mathbf{T}$ denote the unit tangent along C, as

$$\int_C \mathbf{F} \cdot \mathbf{T}\,\mathrm{d}s.$$

But we talked before about how $\mathrm{d}x$, $\mathrm{d}y$ care about the orientation of C, but $\mathrm{d}s$ does not—so what gives?

Computations

Problem 1. Let h be any (nice) single-variable function, and define

$$\mathbf{F} = \left\langle xh(x^2 + y^2), yh(x^2 + y^2)\right\rangle.$$

(a) Show that $\mathbf{F}$ is conservative via differentiation.
(b) Let g denote an antiderivative of h (any nice single-variable function has an antiderivative—even if you can't write it down!). Show that $\mathbf{F}$ is conservative by exhibiting a function f in terms of g such that $\nabla f = \mathbf{F}$.
(c) Let $h(t) = -t^{-3/2}$. The $\mathbf{F}$ that results is very common in physics (though in 3 dimensions rather than 2). However, only one of the preceding two arguments is valid for showing $\mathbf{F}$ is conservative on its entire domain—which one, and why?

Problem 2. Consider the vector field

$$\mathbf{F}(x, y) = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle.$$

(a) Let C be the circle $x^2 + y^2 = 1$ traversed once counterclockwise, starting and ending at $(1, 0)$. Compute

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

(b) Find a function f such that $\nabla f = \mathbf{F}$. How does the domain of your f compare to the domain of $\mathbf{F}$?
(c) Is it possible to find a function f in part (b) that is defined on the entire circle C from part (a)?
(d) Let L be the path that starts at $(1, 1)$, goes in a straight line to $(-2, 0)$, and then goes in another straight line ending at $(3, -3)$. Compute

$$\int_L \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$